UDC 539.3

ASYMPTOTIC ANALYSIS OF THREE-DIMENSIONAL DYNAMIC EQUATIONS OF A THIN PLATE

PMM Vol. 38, №6, 1974, pp. 1072-1078

M. I. GUSEIN-ZADE
(Moscow)

(Received January 24, 1974)

Two-dimensional dynamic equations of thin plate vibrations are obtained from the three-dimensional dynamic equations of elasticity theory on the basis of an asymptotic method [1 – 3]. Such an approach permits establishing the limits of applicability of the two-dimensional dynamic equations and the corresponding boundary and initial conditions, and indicating the means of obtaining refined results.

The question of the construction of an inner state of stress of a thin plate under dynamic conditions is examined herein. The possibility of considering states of stress with distinct variability in time and in the coordinates and with a distinct relationship between the displacement intensities, is taken into account.

Taking account of the presence of mass forces, let us go over to dimensionless quantities in the three-dimensional dynamic equations of elasticity theory in terms of displacements, and let us perform the change of variables

$$v_x = \frac{u_x}{h}\ (xy),\ v_z = \frac{u_z}{h},\ \xi = \varepsilon^{-r}\frac{x}{l}\ (xy),\ \zeta = \frac{z}{h},\ \tau = \frac{t}{t_0} \tag{1}$$

where

$$t_0 = \varepsilon^{\omega}\frac{l^2}{h}\sqrt{\frac{\rho}{E}} = \varepsilon^{\omega-1} l\sqrt{\frac{\rho}{E}} \tag{2}$$

Here $2h$ is the thickness, l is the characteristic dimension of the plate, $\varepsilon = h/l$ is the relative thickness, and t_0 is the characteristic time which it is convenient to represent as (2).

The quantity ω characterizes the variability of the state of stress in time; the larger the ω, the smaller the t_0, and therefore, the more frequently does the process change with time. The quantity $r = p/q$ (p, q are integers) characterizes the variability in the coordinates, where for $r = 0$ ($q = 1$, $p = 0$) the variability is such that the characteristic dimension of the deformation pattern coincides with the characteristic geometric dimension l of the plate.

Let us consider the mass forces X, Y, Z acting on the plate to be constant over its thickness. (If the need to consider the dependence of X, Y, Z on z were to arise, this would then not be difficult).

We seek the solution of the equations obtained from the Lamé equations, after making the passage (1) to the new variables as asymptotic series in the small parameter $\lambda = \varepsilon^{1/q}$

$$v_x = \varepsilon^{\varkappa+1-r}\sum_{s=0}^{\infty}\lambda^s v_x^{(s)}\quad (xy),\qquad v_z = \varepsilon^{\varkappa}\sum_{s=0}^{\infty}\lambda^s v_z^{(s)} \tag{3}$$

where $\varkappa$ is some number which will be determined later.

The equations for $v_x^{(s)}$ (xy), $v_z^{(s)}$ can be integrated with respect to ζ, which yields

$$v_x^{(s)} = \sum_{k=0}^{K+1} \zeta^k v_{xk}^{(s)} \quad (xy), \qquad v_z^{(s)} = \sum_{k=0}^{K} \zeta^k v_{zk}^{(s)} \tag{4}$$

$$K = \begin{cases} [s/q], & \text{if} \quad [s/q] \quad \text{is an even number} \\ [s/q] - 1, & \text{if} \quad [s/q] \quad \text{is an odd number} \end{cases}$$

The square brackets here denote that the integer part of s/q is taken.

We obtain recursion relations for $v_{xk}^{(s)}$ (xy), $v_{zk}^{(s)}$ which permit them to be determined in terms of quantities known from previous approximations ($\delta_k{}^i$ is the Kronecker delta)

$$(k+2)(k+1)\, v_{x,\,k+2}^{(s)} + \frac{k+1}{1-2\nu} v_{z,\,k+1}^{(s)} + \tag{5}$$

$$\frac{1}{1-2\nu}\frac{\partial}{\partial\xi}\left(\frac{\partial v_{xk}^{(s-2q+2p)}}{\partial\xi} + \frac{\partial v_{yk}^{(s-2q+2p)}}{\partial\eta}\right) + \Delta v_{xk}^{(s-2q+2p)} -$$

$$2(1+\nu)\frac{\partial^2 v_{xk}^{(s-4q+2\omega q)}}{\partial\tau^2} + \delta_k{}^0 \delta_s^{-\varkappa q - q + p}(1+\nu)\frac{\rho}{E} X^* = 0 \ (\xi\eta)$$

$$(k+2)(k+1)\, v_{z,\,k+2}^{(s)} + \frac{k+1}{2(1-\nu)}\left(\frac{\partial v_{x,\,k+1}^{(s-2q+2p)}}{\partial\xi} + \frac{\partial v_{y,k+1}^{(s-2q+2p)}}{\partial\eta}\right) +$$

$$\frac{1-2\nu}{2(1-\nu)}\Delta v_{zk}^{(s-2q+2p)} - \frac{(1-2\nu)(1+\nu)}{1-\nu}\frac{\partial^2 v_{zk}^{(s-4q+2\omega q)}}{\partial\tau^2} +$$

$$\delta_k{}^0 \delta_s^{-\varkappa q}\frac{(1-2\nu)(1+\nu)}{2(1-\nu)}\frac{\rho}{E} Z^* = 0$$

$$X^* = 2hX, \quad Y^* = 2hY, \quad Z^* = 2hZ$$

For the stresses we obtain the asymptotic expansions

$$\frac{1}{E}\sigma_{xx} = \varepsilon^{\varkappa}\sum_{s=0}^{\infty}\lambda^s \sigma_{xx}^{(s)} \quad (xy), \qquad \frac{1}{E}\sigma_{xy} = \varepsilon^{\varkappa+2-2r}\sum_{s=0}^{\infty}\lambda^s\sigma_{xy}^{(s)} \tag{6}$$

$$\frac{1}{E}\sigma_{xz} = \varepsilon^{\varkappa+1-r}\sum_{s=0}^{\infty}\lambda^s \sigma_{xz}^{(s)} \quad (xy), \qquad \frac{1}{E}\sigma_{zz} = \varepsilon^{\varkappa}\sum_{s=0}^{\infty}\lambda^s\sigma_{zz}^{(s)} \tag{7}$$

Here

$$\sigma_{xx}^{(s)} = \sum_{k=0}^{K+1}\zeta^k\sigma_{xxk}^{(s)} \ (xy), \qquad \sigma_{xy}^{(s)} = \sum_{k=0}^{K+1}\zeta^k\sigma_{xyk}^{(s)} \tag{8}$$

$$\sigma_{xz}^{(s)} = \sum_{k=0}^{K}\zeta^k\sigma_{xzk}^{(s)} \ (xy), \qquad \sigma_{zz}^{(s)} = \sum_{k=0}^{K+1}\zeta^k\sigma_{zzk}^{(s)}$$

$$\sigma_{xxk}^{(s)} = \frac{\nu}{(1-2\nu)(1+\nu)}(k+1)\, v_{z,\,k+1}^{(s)} + \frac{1-\nu}{(1-2\nu)(1+\nu)}\frac{\partial v_{xk}^{(s-2q+2p)}}{\partial\xi} + \tag{9}$$

$$\frac{\nu}{(1-2\nu)(1-\nu)}\frac{\partial v_{yk}^{(s-2q+2p)}}{\partial\eta} \ (xy), \ \sigma_{xyk}^{(s)} = \frac{1}{2(1+\nu)}\left(\frac{\partial v_{xk}^{(s)}}{\partial\eta} + \frac{\partial v_{yk}^{(s)}}{\partial\xi}\right)$$

$$\sigma_{xzk}^{(s)} = \frac{1}{2(1+\nu)}\left((k+1)\, v_{x,k+1}^{(s)} + \frac{\partial v_{zk}^{(s)}}{\partial\xi}\right) \quad (xy)$$

$$\sigma_{zzk}^{(s)} = \frac{1-\nu}{(1-2\nu)(1+\nu)}(k+1)\, v_{z,k+1}^{(s)} +$$

$$\frac{\nu}{(1-2\nu)(1+\nu)}\left(\frac{\partial v_{xk}^{(s-2q+2p)}}{\partial\xi} + \frac{\partial v_{yk}^{(s-2q+2p)}}{\partial\eta}\right)$$

The boundary conditions

$$\sigma_{xz}(\xi, \eta; \pm 1) = \tau_x^{\pm} \quad (xy), \qquad \sigma_{zz}(\xi, \eta; \pm 1) = \tau_z^{\pm} \tag{10}$$

hold on the face planes of the plate for $\zeta = \pm 1$. Satisfying these conditions, we arrive at a system of equations which describe the internal state of stress of the plate. Let us be interested in those effects on the plate for which its motion is essentially of dynamic character. Hence, in obtaining the equations it must be stipulated that the inertial terms enter into the system of zero approximation equations. Clarifying the possibility of complying with the arbitrary conditions (10) and taking the above into account, we arrive at the deduction that it is necessary, firstly, that the first $2q - 2p$ terms in the expansions (7) for σ_{xz} (xy) and the first $4q - 4p$ terms in the expansions for $\sigma_{zz}^{(s)}$ vanish. This yields

$$v_{x1}^{(s)} + \frac{\partial v_{z0}^{(s)}}{\partial \xi} = 0 \qquad (xy), \quad s < 2q - 2p \tag{11}$$

$$v_{z1}^{(s)} + \frac{\nu}{1-\nu}\left(\frac{\partial v_{x0}^{(s-2q+2p)}}{\partial \xi} + \frac{\partial v_{y0}^{(s-2q+2p)}}{\partial \eta}\right) = 0, \quad s < 4q - 4p$$

Secondly, it is necessary that

$$\omega = 2p/q \quad \text{or} \quad \omega = 2r \tag{12}$$

The relationships (11) are equivalent to compliance with the Kirchhoff-Love hypothesis, and the relationship (12) establishes a relation between the parameters characterizing the variability of the process in time and in the coordinates.

We determine the value of $\varkappa$ in (3) from the condition that the normal surface load is independent of the relative thickness. This yields

$$\varkappa = -4 + 4r \tag{13}$$

Complying with the boundary conditions (10) on the face planes of the plate, we obtain the dynamic equations of a plate in different approximations. For any s they are

$$\frac{1}{1-\nu^2}\frac{\partial}{\partial \xi}\left(\frac{\partial v_{x0}^{(s)}}{\partial \xi} + \frac{\partial v_{y0}^{(s)}}{\partial \eta}\right) + \frac{1}{2(1+\nu)}\frac{\partial}{\partial \eta}\left(\frac{\partial v_{y0}^{(s)}}{\partial \xi} - \frac{\partial v_{x0}^{(s)}}{\partial \eta}\right) = p_x^{(s)} \quad (xy) \tag{14}$$

$$\frac{1}{3(1-\nu^2)}\Delta\Delta v_{z0}^{(s)} + \frac{\partial^2 v_{z0}^{(s)}}{\partial \tau^2} = p_z^{(s)} \tag{15}$$

Introducing the notation

$$\tau_x^+ - \tau_x^- = Q_x \quad (xy), \qquad \tau_z^+ - \tau_z^- = Q_z$$
$$\tau_x^+ + \tau_x^- = M_y \quad (xy), \qquad \tau_z^+ + \tau_z^- = m \tag{16}$$

the right sides of (14) for $s < 4q - 4p$ become

$$p_x^{(s)} = -\delta_s{}^0\frac{\varepsilon^{1-r}}{2E}(Q_x + \rho X^*) - \delta_s^{2q-2p}\frac{1}{2E}\left\{\frac{\nu}{1-\nu}\frac{\partial m}{\partial \xi} + \frac{\varepsilon^{1-r}}{6}\times\right. \tag{17}$$
$$\left[\frac{2+\nu}{1-\nu}\frac{\partial}{\partial \xi}\left(\frac{\partial Q_x}{\partial \xi} + \frac{\partial Q_y}{\partial \eta}\right) - \frac{\partial}{\partial \eta}\left(\frac{\partial Q_x}{\partial \eta} - \frac{\partial Q_y}{\partial \xi}\right) - \right.$$
$$\left.\left.\frac{1-8\nu}{2(1-\nu)}\rho\frac{\partial}{\partial \xi}\left(\frac{\partial X^*}{\partial \xi} + \frac{\partial Y^*}{\partial \eta}\right) + 3\rho\Delta X^*\right]\right\} + \frac{\partial^2 v_{x0}^{(s-2q+2p)}}{\partial \tau^2} \quad (xy)$$

For $s < 6q - 6p$ the right side of (15) becomes

$$p_z^{(s)} = \delta_s{}^0 \frac{1}{2E}\left[Q_z + \varepsilon^{1-r}\left(\frac{\partial M_y}{\partial \xi} + \frac{\partial M_x}{\partial \eta}\right) + \rho Z^*\right] - \tag{18}$$
$$\delta_s^{2q-2p} \frac{1}{20E(1-\nu)} \Delta\left[(8-3\nu) Q_z + \varepsilon^{1-r}\frac{4+\nu}{3}\left(\frac{\partial M_y}{\partial \xi} + \frac{\partial M_x}{\partial \eta}\right) + \right.$$
$$\left.\frac{24+\nu}{3}\rho Z^*\right] + \frac{17-7\nu}{15(1-\nu)}\frac{\partial^2}{\partial \tau^2}\Delta v_{z0}^{(s-2q+2p)} - \delta_s^{4q-4p}\frac{1}{2E(1-\nu)} \times$$
$$\left\{\frac{1}{4200}\Delta\Delta\left[(227-157\nu) Q_z + \varepsilon^{1-r}(87-17\nu)\left(\frac{\partial M_y}{\partial \xi} + \frac{\partial M_x}{\partial \eta}\right) + \right.\right.$$
$$\left.(87-17\nu)\rho Z^*\right] - \frac{1+\nu}{1050}\frac{\partial^2}{\partial \tau^2}\left[3(223-141\nu-22\nu^2) Q_z + \right.$$
$$\varepsilon^{1-r}2(72+101\nu-33\nu^2)\left(\frac{\partial M_y}{\partial \xi} + \frac{\partial M_x}{\partial \eta}\right) +$$
$$\left.\left.2(422-424\nu-33\nu^2)\rho Z^*\right]\right\} - \frac{(1+\nu)(422-424\nu-33\nu^2)}{525(1-\nu)}\frac{\partial^4 v_{z0}^{(s-4q+4p)}}{\partial \tau^4}.$$

Thus, we have an iteration process for the construction of the dynamic internal state of stress of a plate, in each of whose steps it is necessary to solve the same questions. The static equations of the generalized plane state of stress for the problem of plate deformation in its plane (the inertial terms are known from the preceding approximations), and the dynamic equation of classical plate theory for the bending problem. For $s = 0$ the right sides of these equations are expressed in terms of the surface load and the mass forces; for $0 < s < 2q - 2p$ they equal zero, and for $s \geqslant 2q - 2p$ they are expressed in terms of the surface load, the mass forces, and in terms of the sulutions for the preceding approximations.

It is possible to pass from (14), (15) obtained for distinct approximations to equations without an approximation superscript but defined to a given asymptotic accuracy. To the accuracy of $O(\varepsilon^{2-\omega})$, the dynamic equations of a thin plate, written in terms of the initial dimensional qunatities in (1), are

$$\frac{1}{1-\nu^2}\frac{\partial}{\partial x}\left(\frac{\partial u_{x0}}{\partial x} + \frac{\partial u_{y0}}{\partial y}\right) + \frac{1}{2(1+\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u_{y0}}{\partial x} - \frac{\partial u_{x0}}{\partial y}\right) = \tag{19}$$
$$-\frac{1}{2Eh}Q_x \qquad (xy)$$

$$\Delta\Delta u_{z0} + \frac{2\rho h}{D}\frac{\partial^2 u_{z0}}{\partial t^2} = \frac{1}{D}\left[Q_z + h\left(\frac{\partial M_y}{\partial x} + \frac{\partial M_x}{\partial y}\right) + \rho Z^*\right] \tag{20}$$
$$D = 2Eh^3/3(1-\nu^2)$$

where u_{x0}, u_{y0}, u_{z0} are the displacements of points of the middle surface. Therefore, we obtain the same internal state of stress equation for dynamic processes with different variability in time (i.e. for different ω). The static equations of the generalized plane state of stress and the dynamic transverse vibrations equation which is not hyperbolic. However, the accuracy of these equations depends substantially on ω. As ω increases the accuracy diminishes and for $\omega = 2$ the equations become entirely inaccurate.

As follows from (1), (3) and (12), for $\omega = 2$ the characteristic dimension of the deformation pattern for a plate becomes equal to h and the intensities of all the displacements are identical. This indicates the essentially three-dimensional nature of the process at $\omega = 2$.

More exact dynamic equations of a plate can be obtained from (14), (15). Equations to the accuracy $O(\varepsilon^{4-2\omega})$ are

$$\frac{1}{1-\nu^2}\frac{\partial}{\partial x}\left(\frac{\partial u_{x0}}{\partial x}+\frac{\partial u_{y0}}{\partial y}\right)+\frac{1}{2(1+\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u_{y0}}{\partial x}-\frac{\partial u_{x0}}{\partial y}\right)-\tag{21}$$
$$\frac{\rho}{E}\frac{\partial^2 u_{x0}}{\partial t^2}=\frac{1}{2Eh}\left\{-Q_x-h\frac{\nu}{1-\nu}\frac{\partial m}{\partial x}-\rho X^*-\right.$$
$$\frac{h^2}{6}\left[\frac{2+\nu}{1-\nu}\frac{\partial}{\partial x}\left(\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial Q_x}{\partial y}-\frac{\partial Q_y}{\partial x}\right)-\right.$$
$$\left.\left.\frac{1-8\nu}{2(1-\nu)}\rho\frac{\partial}{\partial x}\left(\frac{\partial X^*}{\partial x}+\frac{\partial Y^*}{\partial y}\right)+3\rho\Delta X^*\right]\right\}\qquad (xy)$$

$$\Delta\Delta u_{z0}-\frac{2\rho h^3}{D}\frac{17-7\nu}{15(1-\nu)}\frac{\partial^2}{\partial t^2}\Delta u_{z0}+\frac{2\rho h}{D}\frac{\partial^2 u_{z0}}{\partial t^2}=\tag{22}$$
$$\frac{1}{D}\left\{Q_z+\rho Z^*+h\left(\frac{\partial M_y}{\partial x}+\frac{\partial M_x}{\partial y}\right)-h^2\frac{1}{20E(1-\nu)}\times\right.$$
$$\left.\Delta\left[(8-3\nu)Q_z+\frac{4+\nu}{3}h\left(\frac{\partial M_y}{\partial x}+\frac{\partial M_x}{\partial y}\right)+\frac{24+\nu}{3}\rho Z^*\right]\right\}$$

It is seen from (21) that to $O(\varepsilon^{4-2\omega})$ accuracy the generalized plane stress equations are of wave nature. All the terms needed to assure $O(\varepsilon^{4-2\omega})$ accuracy are retained in (22) which describes the transverse plate motion. This equation is not completely hyperbolic. A certain wave process associated with the presence of a term with $\partial^2\Delta u_{z0}/\partial t^2$ is imposed on the transverse plate motion and instantaneously encompasses the whole plate.

Meanwhile, a Timoshenko-type equation for a plate obtained by Ufliand [4], taking into account the shear caused by transverse stresses and rotational inertia, is completely hyperbolic. This is related to the fact that a term with $\partial^4 u_{z0}/\partial t^4$ was taken into account in the equation mentioned; however, as an asymptotic analysis shows, it should be taken into account only in equations with a higher asymptotic accuracy.

The equation of transverse plate motion obtained by an asymptotic method and having $O(\varepsilon^{6-3\omega})$ accuracy is

$$\Delta\Delta u_{z0}-\frac{2\rho h^3}{D}\frac{17-7\nu}{15(1-\nu)}\frac{\partial^2}{\partial t^2}\Delta u_{z0}+\frac{2\rho^2 h^3}{ED}\frac{(1+\nu)(422-424\nu-33\nu^2)}{525(1-\nu)}\times\tag{23}$$
$$\frac{\partial^4 u_{z0}}{\partial t^4}+\frac{2\rho h}{D}\frac{\partial^2 u_{z0}}{\partial t^2}=\frac{1}{D}\left\{Q_z+h\left(\frac{\partial M_y}{\partial x}+\frac{\partial M_x}{\partial y}\right)+\rho Z^*-\right.$$
$$h^2\frac{1}{20E(1-\nu)}\Delta\left[(8-3\nu)Q_z+h\frac{4+\nu}{3}\left(\frac{\partial M_y}{\partial x}+\frac{\partial M_x}{\partial y}\right)+\right.$$
$$\left.\frac{24+\nu}{3}\rho Z^*\right]-h^4\frac{1}{2E(1-\nu)}\times$$
$$\left\{\frac{1}{4200}\Delta\Delta\left[(227-157\nu)Q_z+h(87-17\nu)\left(\frac{\partial M_y}{\partial x}+\frac{\partial M_x}{\partial y}\right)+\right.\right.$$
$$\left.(87-17\nu)\rho Z^*\right]-\frac{1+\nu}{1050}\frac{\partial^2}{\partial t^2}\left[3(223-141\nu-22\nu^2)\,Q_z+\right.$$
$$h2(72+101\nu-33\nu^2)\left(\frac{\partial M_y}{\partial x}+\frac{\partial M_x}{\partial y}\right)+$$
$$\left.\left.\left.2(422-424\nu-33\nu^2)\rho Z^*\right]\right\}\right\}$$

Besides the shear strains and rotational inertia, finer factors are also taken into account in this equation, whose physical meaning is difficult to discover. As the Ufliand equation [4], this equation is of completely hyperbolic type, but differs in the coefficients and the right sides.

Let us investigate what type the equations of transverse plate motion are, corresponding to still higher accuracy. Let us write down just the higher terms governing the kind of equation. We have

$$d_1 \frac{\partial^4}{\partial t^4} \Delta u_{z0} + \ldots = \ldots \qquad (O(\varepsilon^{8-4\omega})) \tag{24}$$

$$d_1 \frac{\partial^4}{\partial t^4} \Delta u_{z0} + d_2 \frac{\partial^6 u_{z0}}{\partial t^6} + \ldots = \ldots \qquad (O(\varepsilon^{10-5\omega}))$$

Thus, the equations of transverse plate motions determined with a higher accuracy than $O(\varepsilon^{6-3\omega})$, do not belong to the completely hyperbolic type. As the accuracy of the equations increases, terms appear with higher and higher orders of the time derivatives; however, the coefficients of these derivatives diminish as the order of these derivatives grows.

It follows from the above that to obtain refined results for the dynamic behavior of a thin plate, it is expedient to use an iteration process whose construction reduces to the solution of the same usual equations (14), (15) at each step, i.e. to the solution of the static equations of the generalized plane state of stress and the dynamic bending equation of the classical theory.

The refined equations corresponding to diverse asymptotic accuracy are constructed for a comparison with the refined equations obtained on the basis of hypotheses, and for estimation of their accuracy.

Depending on the value of the parameter ω characterizing the variability of the state of stress in time, the following classification of the dynamic processes occurring in a thin plate can be proposed.

(1) Quasistatic processes $(\omega < 0)$. The two-dimensional equations for a number of first approximations do not contain inertial terms (the time t is considered a parameter). In the equations of those approximations in which the inertial terms appear, they are in the right sides and are known from previous approximations.

(2) Dynamic processes $(0 \leqslant \omega < 2)$. The problem of deformation of a plate in its plane is of quasistatic character and the bending problem reduces to the dynamic equation of classical theory.

(3) Essentially three-dimensional processes $(\omega \geqslant 2)$ which cannot be described by any two-dimensional theories.

Let us still examine one more question. We note that the velocity of longitudinal wave propagation equals $\sqrt{E/\rho}$ for a medium with the Poisson's ratio $\nu = 0$ and $1.16\sqrt{E/\rho}$ for a medium with $\nu = 0.3$. Hence, the quantity $\sqrt{E/\rho}$ is close to the propagation velocity of perturbations in real materials. The quantity $L(t_0) = t_0\sqrt{E/\rho}$ is commensurate with the distance to which a perturbation is propagated in the plate material in the characteristic time t_0. We have

$$l\varepsilon^{-1} \leqslant L(t_0) < h$$

for dynamic processes for which $0 \leqslant \omega < 2$. Therefore, for dynamic processes for which the characteristic dimension of the deformation pattern is commensurate with the characteristic geometric dimension l (i.e. for $\omega = 0$), the perturbation traverses a distance ε^{-1}-fold greater than the characteristic plate dimension in the time t_0. As ω increases, the distance $L(t_0)$ diminishes. For $\omega = 1$ when the characteristic dimension of the deformation pattern is commensurate with $\sqrt{lh}$, the quantity $L(t_0)$ is commensurate with l. But for $\omega = 2$ the characteristic dimension of the deformation pattern and the quantity $L(t_0)$ become commensurate with h. This also indicates that the dynamic processes corresponding to the values $\omega \geqslant 2$ are essentially three-dimensional in nature.

REFERENCES

1. Gol'denveizer, A. L., Derivation of an approximate theory of bending of a plate by the method of asymptotic integration of the equations of the theory of elasticity. PMM Vol. 26, №4, 1962.
2. Gol'denveizer, A. L. and Kolos, A. V., On the two-dimensional equations in the theory of thin elastic plates. PMM Vol. 29, №1, 1965.
3. Gusein-Zade, M. I., On some properties of the state of stress of a thin elastic layer. PMM Vol. 31, №6, 1967.
4. Ufliand, Ia. S., Wave propagation with transverse vibrations of bars and plates. PMM Vol. 12, №3, 1948.

Translated by M. D. F.

UDC 539.3

HERTZ PROBLEM ON COMPRESSION OF ANISOTROPIC BODIES

PMM Vol. 38, №6, 1974, pp. 1079-1083

V. A. SVEKLO
(Kaliningrad)

(Received February 18, 1974)

On the basis of results in [1], a derivation is given of the fundamental Hertz relationships for the compression of anisotropic (orthotropic) bodies which differs from [2]. It is shown that if the elastic constants satisfy some additional conditions, then the domain of contact is a circle in the compression of axisymmetric bodies along their common axes of geometric symmetry.

1. Formulation of the problem and its solution. Two bodies initially touching at a point and subjected to compressive forces P have a common elliptical contact area after deformation because of its smallness. If z_1 and z_2 are in the same direction as the internal normals to the surfaces bounding the bodies at the point of their initial contact, then the x, y axes in the common tangent plane can always be selected so that the equality

$$w_1 + w_2 = \delta - x^2 / 2R_1 - y^2 / 2R_2 \qquad (1.1)$$

would hold in the contact domain. Here w_j are elastic displacements of the body points in the z_j direction, δ is the approach of the bodies, R_j are specified and determined by the shape of the bodies in the neighborhood of their initial contact point [1]. The pressure domains of the bodies are replaced by half-spaces in the computation of w_j because of the smallness of the dimensions. Therefore, in conformity with [1], the stress on the pressure area is determined by

$$\sigma_z = 3P\,(2\pi ab)^{-1}\,(1 - x^2 / a^2 - y^2 / b^2) \qquad (1.2)$$